Final Review

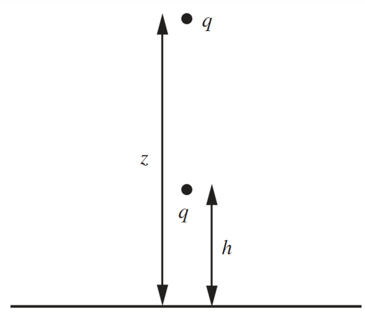
A Puzzle...

Direction of the Force

Example

A point charge q is located a fixed height h above an infinite horizontal conducting plane. Another point charge q is located a height z (with $z > h$) above the plane. The two charges lies on the same vertical line. If z is only slightly larger than h, then the force on the top charge is clearly upward. But for larger values of z, is the force still always upward?

Hint: Try to solve this without doing any calculations. Think dipole.



Solution

There are two negative image charges on the other side of the plane, at the mirror-image locations. For very large z values of the top charge q, the lower q and its image charge $-q$ look like a dipole from afar, which has a repulsive (upward) field that falls off like $\frac{1}{z^3}$. But the attractive (downward) field from the other image charge $-q$ behaves like $\frac{1}{(2z)^2}$. This has a smaller power of z in the denominator, so it dominates for large z. The force on the top charge q is therefore downward for large z. So the answer to the stated question is "No." □

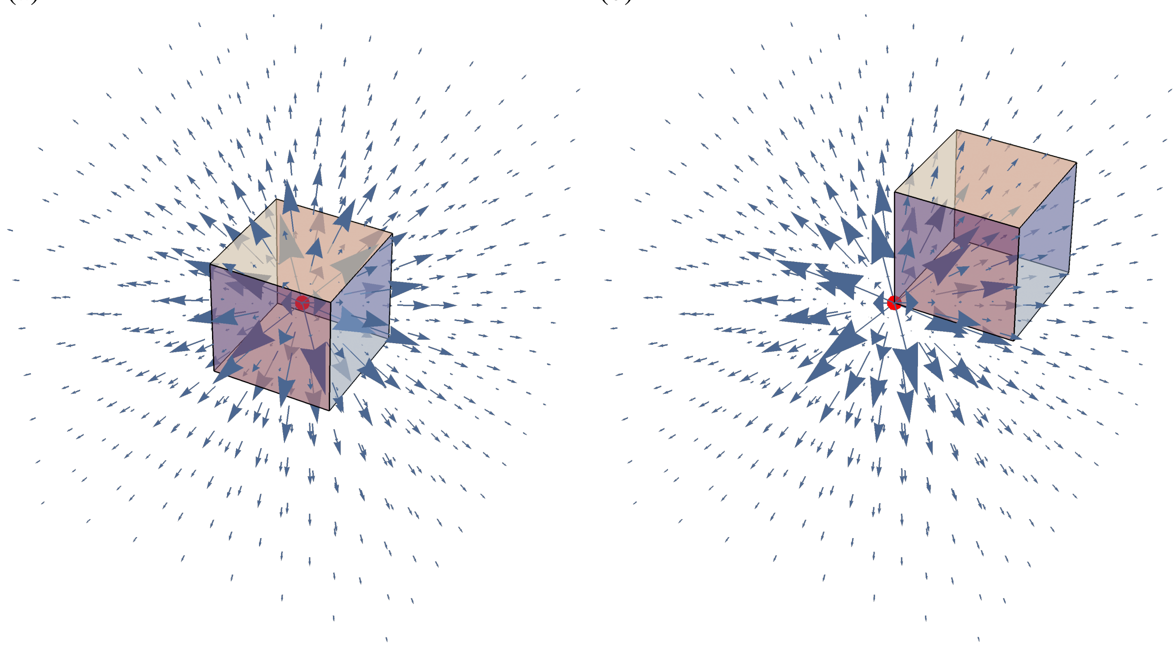
Flux through a Cube

Example

(a) A point charge q is located at the center of a cube of edge d. What is the value of $\int \vec{E} \cdot d\vec{a}$ over one face of the cube?

(b) The charge q is moved to one corner of the cube. Now what is the value of the flux of $\vec{E}$ through each of the faces of the cube? (To make things well defined, treat the charge like a tiny sphere.)

(a) (b)



Solution

(a) Since the total flux through the cube must equal $\frac{q}{\epsilon_0}$, by symmetry the flux through one face of the cube must be $\frac{q}{6\,\epsilon_0}$.

We could, of course, double check this by direct integration. Let's find the flux through the top face of the cube with vertices $\left(\pm\frac{a}{2}, \pm\frac{a}{2}, \frac{a}{2}\right)$. The magnitude of the electric field equals

$$E = \frac{k\,q}{x^2+y^2+\left(\frac{a}{2}\right)^2} \tag{1}$$

and the area element is

$$d\vec{a} = d\,a\,\hat{z} \tag{2}$$

The component E_z (which is the only one that survives in $\vec{E}\cdot d\vec{a}$) equals

$$E_z = \frac{k\,q}{x^2+y^2+\left(\frac{a}{2}\right)^2}\,\frac{\frac{a}{2}}{\left(x^2+y^2+\left(\frac{a}{2}\right)^2\right)^{1/2}} \tag{3}$$

Therefore, the flux going out through the cube's top face equals

$$\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\frac{k\,q}{x^2+y^2+\left(\frac{a}{2}\right)^2}\,\frac{\frac{a}{2}}{\left(x^2+y^2+\left(\frac{a}{2}\right)^2\right)^{1/2}}\,dx\,dy = \frac{q}{6\,\epsilon_0} \tag{4}$$

$$\text{Integrate}\left[\frac{k\,Q}{\left(x^2+y^2+\left(\frac{a}{2}\right)^2\right)}\,\frac{\frac{a}{2}}{\left(x^2+y^2+\left(\frac{a}{2}\right)^2\right)^{1/2}},\ \left\{x,-\frac{a}{2},\frac{a}{2}\right\},\ \left\{y,-\frac{a}{2},\frac{a}{2}\right\},\ \text{Assumptions}\to 0<a\right]\ /.\ k\to\frac{1}{4\,\pi\,\epsilon0}$$

$$\frac{Q}{6\,\epsilon0}$$

as expected.

(b) In this case, the flux through the three closest face to the charge will have zero flux going through them (because the electric field points *along* these faces, so that $\vec{E}$ and $d\vec{a}$ are perpendicular). The three remaining faces are all symmetric with respect to the point charge, and therefore each of them must have the same flux outwards. Since $\frac{1}{8}$ of the charge is enclosed by the cube, we can model the point charge as a spherical charge with radius R

where R is very small. Thus we expect a total flux of $\frac{q}{8\,\epsilon_0}$ through the cube, which implies a flux of $\frac{q}{24\,\epsilon_0}$ through each of the three far faces of the cube.

Again, we can compute this explicitly. Let's integrate along the top face of the cube with vertices $(0, 0, a)$, $(a, 0, a)$, $(0, a, a)$, and (a, a, a). The magnitude of the electric field and area elements equal

$$E = \frac{k\,q}{x^2+y^2+a^2} \tag{5}$$

$$d\vec{a} = da\,\hat{z} \tag{6}$$

The component E_z (which is the only one that survives in $\vec{E}\cdot d\vec{a}$) equals

$$E_z = \frac{k\,q}{x^2+y^2+a^2}\,\frac{a}{\left(x^2+y^2+a^2\right)^{1/2}} \tag{7}$$

Therefore the flux through this face equals

$$\int_0^a \int_0^a \frac{k\,q}{x^2+y^2+a^2}\,\frac{a}{\left(x^2+y^2+a^2\right)^{1/2}}\,dx\,dy = \frac{q}{24\,\epsilon_0} \tag{8}$$

```
Integrate[kQ/(x^2 + y^2 + a^2) a/(x^2 + y^2 + a^2)^(1/2), {x, 0, a}, {y, 0, a}, Assumptions → 0 < a] /. k → 1/(4 π ϵ0)
```

```
Q/(24 ϵ0)
```

Note that if the charge were a true point charge, and if it were located just inside or just outside the cube, then the field would *not* be parallel to each of the three faces that touch the given corner. The flux would depend critically on the exact location of the point charge. Replacing the point charge with a small sphere, whose center lies at the corner, eliminates this ambiguity. □

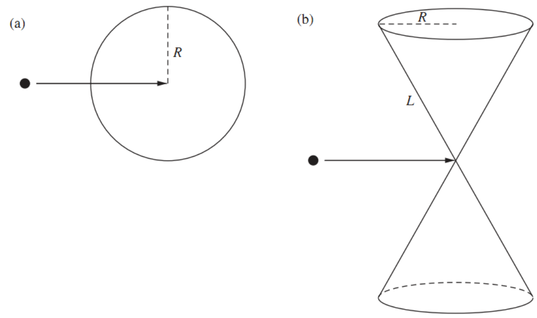
Moving at the Speed of Light

One of the interesting quirks about the velocity addition formula is that if you start off moving at c in one frame, then you move in c in another frame. This begs some interesting questions, such as what happens if you accelerate a car to the speed of light, and you turn on your headlights. Would the light move at speed c relative to you, would it all pool inside of the headlight, or would something altogether different happen? Michael Stevens has an amazing YouTube video analyzing this very question.

Electrostatics

Sphere and Cones

<u>Example</u>
(a) Consider a fixed hollow spherical shell with radius R and surface charge density σ. A particle with mass m and charge $-q$ that is initially at rest falls in from infinity. What is its speed when it reaches the center of the shell? (Assume that a tiny hole has been cut in the shell, to let the charge through.)
(b) Consider two fixed hollow conical shells (that is, ice cream cones without the ice cream) with base radius R, slant height L, and surface charge density σ, arranged as shown in figure (b) below. A particle with mass m and charge $-q$ that is initially at rest falls in from infinity, along the perpendicular bisector line, as shown. What is its speed when it reaches the tip of the cones?
(The answers to both parts of this problem should relate very nicely!)



<u>Solutions</u>

(a) This problem is terrifically simple if we use the electric potential. Recall that for a spherical shell, the potential equals $\phi[r] = \frac{k Q}{r}$ where $Q = 4 \pi R^2 \sigma$ for $r \geq R$. Inside the sphere, the electric field is zero so that the potential must be a constant; by continuity, the electric potential inside the shell equals $\phi[r] = \frac{k Q}{R}$ for $r \leq R$.

Therefore the work required to bring a particle from ∞ to the center of the spherical shell equals $-q\,\phi[0]$, so that the particles energy due to free fall would be $\frac{1}{2} m v^2 = q\,\phi[0] = \frac{k q Q}{R}$ or equivalently

$$v = \left(\frac{2 k q Q}{m R}\right)^{1/2} = \left(\frac{2 q R \sigma}{m \epsilon_0}\right)^{1/2} \tag{9}$$

We could also do this (more painfully) by integrating the electric field and solving the corresponding differential equation. The electric field for $r \geq R$ equals $\vec{E} = \frac{k Q}{r^2} \hat{r}$ so that the force accelerating the particle radially inward satisfies

$$m v \frac{dv}{dr} = m \ddot{r} = -\frac{k Q q}{r^2} \tag{10}$$

where we have used the relation $\frac{d^2 r}{dt^2} = \ddot{r} = \frac{d\dot{r}}{dr} \frac{dr}{dv} = \frac{dv}{dr} v$. Separating the variables and integrating t differential equation above,

$$\int_0^v m\, v\, dv = \int_\infty^r -\frac{k Q q}{r^2}\, dr \tag{11}$$

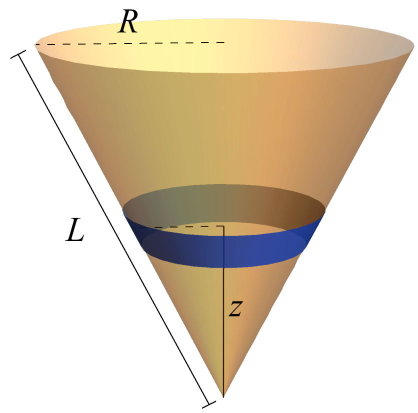
$$\frac{1}{2} m v^2 = \frac{k Q q}{r} \tag{12}$$

This formula is valid from $r = \infty$ until $r = R$, at which point the electric field becomes zero and the velocity remains constant until the particle hits the center, moving at speed

$$\frac{1}{2} m v^2 = \frac{k Q q}{R} \tag{13}$$

The solution proceeds as above.

(b) Since cones may seem like foreign objects, let start by first integrating over the surface area of the cone to make sure that we correctly find its surface area (the surface area of a cone is $\pi R L$, as per Wikipedia). We will integrate in rings from $z = 0$ to $z = H = \sqrt{L^2 - R^2}$, as shown below.



The surface area of a ring between z and $z + dz$ is given by

$$\text{(surface area of ring between } z \text{ and } z + dz) = 2\pi \,\text{(radius of ring) (width of ring)} \tag{14}$$

Noting the similar triangles, the radius of a ring at height z equals $z\left(\frac{R}{H}\right)$. The vertical height of the ring is dz, but we want to know the slant width (i.e. the diagonal length) of the ribbon. Since the ring goes vertically up by dz and outward by $\frac{R}{H} dz$, the slant width is $\left(1 + \left(\frac{R}{H}\right)^2\right)^{1/2} dz$. Thus

$$\text{(surface area of ring between } z \text{ and } z + dz) = 2\pi z \left(\frac{R}{H}\right)\left(1 + \left(\frac{R}{H}\right)^2\right)^{1/2} dz \tag{15}$$

and the surface area of the entire cone equals

$$\begin{aligned} \text{surface area} &= \int_0^H 2\pi z \left(\frac{R}{H}\right)\left(1 + \left(\frac{R}{H}\right)^2\right)^{1/2} dz \\ &= \pi R \left(H^2 + R^2\right)^{1/2} \\ &= \pi R L \end{aligned} \tag{16}$$

as desired. Having confirmed this neat fact, let us return to the problem. The area element of a cone is given by Equation (16), since we just showed that this integral covers the full surface of the cone and yields the correct surface area formula. The potential at the base of the two cones, $\phi[0]$, equals twice the contribution from a single cone, written in the familiar form $\int \frac{k\,dq}{r}$, equals

$$\begin{aligned} \phi[0] &= 2 \int_0^H \frac{k\left(2\pi z \left(\frac{R}{H}\right)\left(1+\left(\frac{R}{H}\right)^2\right)^{1/2} dz\right)\sigma}{z\left(1+\left(\frac{R}{H}\right)^2\right)^{1/2}} \\ &= 4\pi k \sigma \left(\frac{R}{H}\right) \int_0^H dz \\ &= \frac{\sigma R}{\epsilon_0} \end{aligned} \tag{17}$$

Since this has the exact same form as the potential $\phi[0] = \frac{k\,Q}{R} = \frac{\sigma R}{\epsilon_0}$ we found in Part (a), the solution proceeds as above, and we once again find the same velocity $v = \left(\frac{2\,q\,R\,\sigma}{\epsilon_0}\right)^{1/2}$. What a remarkable coincidence! □

Conductors

Image Charges for Two Planes

This section investigates the following problem very deeply!

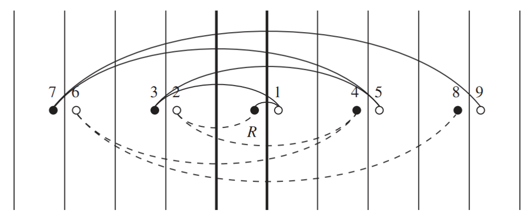
Example

A point charge q is located between two parallel infinite conducting planes, a distance b from one and $l - b$ from the other. Where should image charges be located so that the electric field is everywhere perpendicular to the planes?

Solution

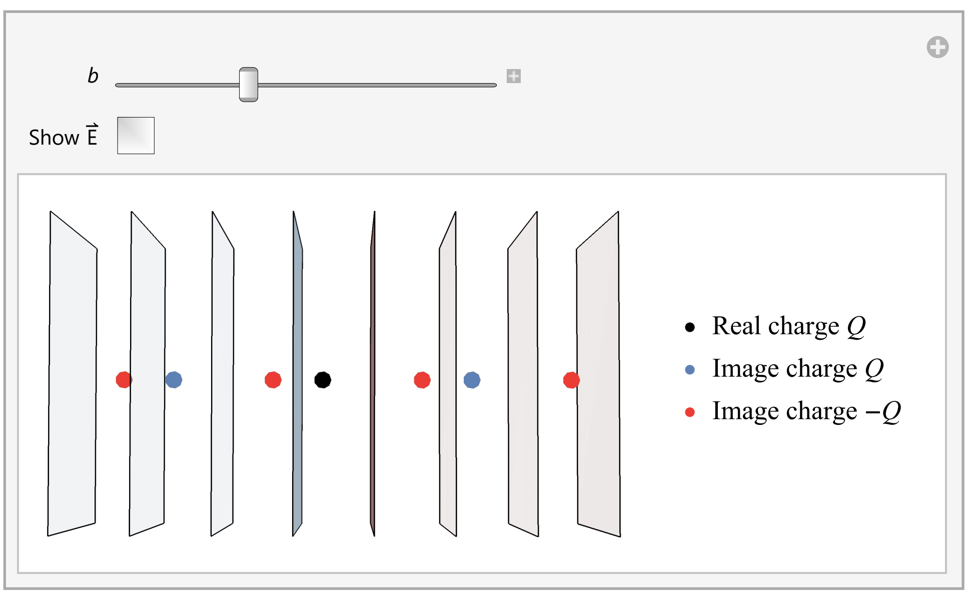
Set one plate at $z = 0$ and one at $z = l$, and let the positron be at $z = b$. The problem is clearly one dimensional, so we fix $x = y = 0$ for all points we discuss. If we just consider the bottom plate, we can place a negative charge at $-b$, but then we need to take care of both of these charges with images beyond the top plate with a negative charge at $z = 2\,l - b$ and a positive charge at $z = 2\,l + b$. In other words, there will be a cascade of image charges (of alternating signs!)

In the figure below, the two given planes are indicated by the bold lines, and the given real charge is labeled R. It turns out that we will need an infinite number of image charges, as shown. Solid dots are positive, hollow dots are negative (assuming the given real charge is positive).



The following Manipulate shows the real charge q (in black) together with the positive image charges (blue) and negative image charges (orange). The real conducting planes are in the middle and are darker than the other planes at $z = -l$, $\pm 2\,l$, $\pm 3\,l$ ...

Out[ ]=



The pattern clearly emerges:

Positive Charges	Negative Charges
...	...
$4\,l+b$	$4\,l-b$
$2\,l+b$	$2\,l-b$
$0\,l+b$	$0\,l-b$
$-2\,l+b$	$-2\,l-b$
$-4\,l+b$	$-4\,l-b$
...	...

If you want, you can group the charges into two sets – the odds and evens, as indicated by the connecting lines in the figure above. Each odd charge corrects the effect of the previous odd charge, with respect to alternating planes. Likewise for the evens.

In the special case where the given real charge is located midway between the two planes, all the image charges are similarly located midway between the (imaginary) planes in the figure above. So the net force on the given charge is zero, as it should be. □

The Limit $b \ll l$

<u>Example</u>

The general force on the real charge is complicated, but it is tractable in the limit $b \ll l$ when the charge q is very close to one of the plates. Find an approximate expression for the force on the charge in this limit.

<u>Solution</u>

To zeroth order, the force on the charge q comes solely from the image charge nearby at $z = -b$, which is a force

$$F \approx \frac{k\,q^2}{(2\,b)^2} \tag{19}$$

downwards. But we can do much better than that! All the remaining pairs of charges can be approximated as dipoles; two of these dipoles are at a distance $2\,l$, another two are at $4\,l$, and so on. Recall the formula $\frac{2\,k\,q\,d}{r^3}$ for the electric field of a dipole along its axis (where d is the distance between the two charges and r is the distance from the dipole's center). Note that every dipole pushes the charge q in the $+z$-direction, the total downwards force on the dipole equals

$$\begin{aligned} F &= \frac{k\,q^2}{(2\,b)^2} - 2\,\{2\,k\,q^2\,(2\,b)\}\left(\frac{1}{(2\,l)^3} + \frac{1}{(4\,l)^3} + \frac{1}{(6\,l)^3} + \cdots\right) \\ &= \frac{k\,q^2}{4\,b^2} - \frac{k\,q^2\,b}{l^3}\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots\right) \end{aligned} \tag{20}$$

where the first factor of 2 in the second term comes from the fact that there are two pairs of dipoles at distance $2\,l$, $4\,l$ ... (on the left and right side of the charge). The factor in parenthesis equals Zeta[3] ≈ 1.202

```
Sum[1/n^3, {n, 1, ∞}]
```

```
Zeta[3]
```

You can also calculate the total force by looking at the forces from the positive and negative image charges separately. From the figure above, the force on the real charge q from the other positive charges will always be 0 by symmetry, but the downwards force from the negative charges equals

$$F = \frac{k q^2}{(2b)^2} - k q^2 \sum_{n=1}^{\infty} \left(\frac{1}{(2 n l - 2 b)^2} - \frac{1}{(2 n l + 2 b)^2} \right)$$
$$= \frac{k q^2}{4 b^2} - \frac{k q^2}{4 l^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{1}{\left(1 - \frac{b}{n l}\right)^2} - \frac{1}{\left(1 + \frac{b}{n l}\right)^2} \right)$$
$$\approx \frac{k q^2}{4 b^2} - \frac{k q^2}{4 l^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\left\{1 + \frac{2b}{n l}\right\} - \left\{1 - \frac{2b}{n l}\right\} \right)$$
$$= \frac{k q^2}{4 b^2} - \frac{k q^2 b}{l^3} \sum_{n=1}^{\infty} \frac{1}{n^3}$$

where in the third step we used the Taylor series $\frac{1}{(1+\epsilon)^2} \approx 1 - 2\epsilon + O[\epsilon]^2$. This result matches equation (20), as expected. □

Advanced Section: The General Solution

Example

Compute the energy of the charge q when it is a distance b and $l - b$ from the two infinite conducting planes. Take the point midway between the two plates (when $b = \frac{l}{2}$) as the zero energy of energy, and compute the work required to move the particle from that point. *Hint*: Use *Mathematica* copiously!

Solution

From above, the force from the positive charges cancels and the upwards force from the negative charges equals

$$F = -\frac{k q^2}{(2b)^2} + k q^2 \sum_{n=1}^{\infty} \left(\frac{1}{(2 n l - 2 b)^2} - \frac{1}{(2 n l + 2 b)^2} \right)$$
$$= -\frac{k q^2}{4 b^2} + \frac{k q^2}{4 l^2} \left(\text{PolyGamma}\left[1, 1 - \frac{b}{l}\right] - \text{PolyGamma}\left[1, 1 + \frac{b}{l}\right] \right) \quad (22)$$

In[∘]:=
```
F = FullSimplify[-k q^2/(2 b)^2 + k q^2 Sum[1/(2 n l - 2 b)^2 - 1/(2 n l + 2 b)^2, {n, 1, ∞}], Assumptions → 0 < b < l]
```

Out[∘]=
$$\frac{1}{4} k q^2 \left(-\frac{1}{b^2} + \frac{\text{PolyGamma}\left[1, 1 - \frac{b}{l}\right] - \text{PolyGamma}\left[1, \frac{b+l}{l}\right]}{l^2} \right)$$

where we have invoked our *Mathematica* superpowers. As an important double-check, we can consider the limit $b \ll l$ and verify that we recoup Equation (20),

$$F_{b \ll l} \approx \frac{k q^2}{4 b^2} + \frac{k q^2 b}{l^3} \frac{\text{PolyGamma}[2,1]}{2} \quad (23)$$

```
Series[F, {b, 0, 2}]
```

$$\frac{k q^2}{4 b^2} + \frac{k q^2 \, \text{PolyGamma}[2, 1] \, b}{2 l^3} + O[b]^3$$

Comparing this to equation (20), we see that they would match if $\frac{\text{PolyGamma}[2,1]}{2} = -\text{Zeta}[3]$, which we can easily check is exactly true,

```
FullSimplify[PolyGamma[2, 1] == -2 Zeta[3]]
```

```
True
```

See, there is no reason to be afraid of scary named functions with *Mathematica* at your back! Fortified by this knowledge, we can now calculate the energy as

$$E = \int_{\frac{l}{2}}^{b} F \, dz = -\frac{k q^2}{4 b} + \frac{k q^2}{4 l} \left(\text{HarmonicNumber}\left[-\frac{b}{l}\right] + \text{HarmonicNumber}\left[\frac{b}{l}\right] + \text{Log}[16] \right) \quad (24)$$

```
FullSimplify@Integrate[F, {b, 1/2, bb}, Assumptions → 0 < bb < l/2] /. bb → b
```

$$\frac{1}{4\,b\,l} k\,q^2 \left(-l + b \left(\text{HarmonicNumber}\left[-\frac{b}{l}\right] + \text{HarmonicNumber}\left[\frac{b}{l}\right] + \text{Log}[16]\right)\right)$$

How yummy! But if we agree to fix l and only move b around, then we can ignore the Log[16] term (i.e. subtract it from all potentials) and get the equivalent energy

$$\tilde{E} = -\frac{k\,q^2}{4\,b} + \frac{k\,q^2}{4\,l}\left(\text{HarmonicNumber}\left[-\frac{b}{l}\right] + \text{HarmonicNumber}\left[\frac{b}{l}\right]\right) \tag{25}$$

Victory! But, wait, there's more!!! We can double check this result by computing the energy of the final configuration assuming that all the image charges are semi-real; in other words, we consider the interaction between every image charge and the real charge, but not between two image charges. This potential (at the image charge) equals

$$\begin{aligned}\phi &= -\frac{k\,q}{2\,b} + k\,q \sum_{n=1}^{\infty}\left(\frac{2}{2\,n\,l} - \frac{1}{2\,n\,l-2\,b} - \frac{1}{2\,n\,l+2\,b}\right) \\ &= -\frac{k\,q}{2\,b} + \frac{k\,q}{2\,l}\left(\text{HarmonicNumber}\left[-\frac{b}{l}\right] + \text{HarmonicNumber}\left[\frac{b}{l}\right]\right)\end{aligned} \tag{26}$$

```
FullSimplify[-kq/(2b) + kq Sum[1/(n l) - 1/(2 n l - 2 b) - 1/(2 n l + 2 b), {n, 1, ∞}]]
```

$$\frac{1}{2\,b\,l} k\,q \left(-l + b \left(\text{HarmonicNumber}\left[-\frac{b}{l}\right] + \text{HarmonicNumber}\left[\frac{b}{l}\right]\right)\right)$$

Therefore, we find

$$\tilde{E} = \frac{1}{2}\,\phi\,q \tag{27}$$

That's freaking cool, and it also sounds very familiar. The exact same thing happens when we considered a single conducting plate (i.e. the classic image charge problem), and it also happened during the spherical conducting plate image charge problem. What is the source of this magical $\frac{1}{2}$? We continue letting ϕ denote the potential energy of the final configuration *assuming that the image charges are semi-real* (so that only interactions between the image charges and the real charge matter, and that interactions between image charges are irrelevant). We would like to show that the energy of the system is always $\frac{1}{2}\,\phi\,q$ for an image problem (which is any problem where you bring in a charge near a grounded conductor, with your charge being the only real charge present).

Suppose that the real charge was placed in its final location, and the magnitude of its charge can be tuned as $\lambda\,q$ where $\lambda \in [0, 1]$. We will tune λ from 0 to 1. For any λ, the point charge has charge $\lambda\,q$, so all of the image charges that it induces are also reduced by the fraction λ of their final values, which implies that the electrostatic potential of the image charges is also reduced to λ of its value. If ϕ is the final value of that potential, the total work performed in raising λ from 0 to 1 equals

$$\int_0^1 (q\,d\lambda)\,\lambda\,\phi = \frac{1}{2}\,q\,\phi \tag{28}$$

This is the source of the $\frac{1}{2}$ factor.

Advanced Section: Charge on Each Plane

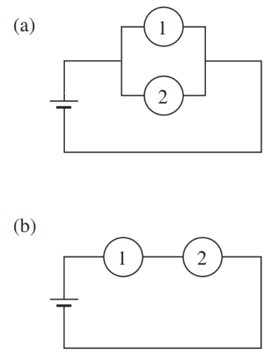
Circuits

Two Light Bulbs

<u>Example</u>

Certain light bulbs can be treated as resistors, with the brightness of the bulb proportional to the power dissipated in the bulb's resistor.

(a) Two light bulbs are connected in parallel, and then connected to a battery, as shown in Panel (a). You observe that bulb 1 is twice as bright as bulb 2. Which bulb's resistor is larger, and by what factor?
(b) The bulbs are now connected in series, as shown in Panel (b). Which bulb is brighter, and by what factor? How bright is each bulb compared with bulb 1 in Part (a)?



Solution
(a) The power dissipated takes the form $\frac{V^2}{R}$. Both bulbs have the same voltage drop V, so if bulb 1 is twice as bring as bulb 2, it must have half the resistance, $R_2 = 2\,R_1$. In parallel, the larger resistor is dimmer.

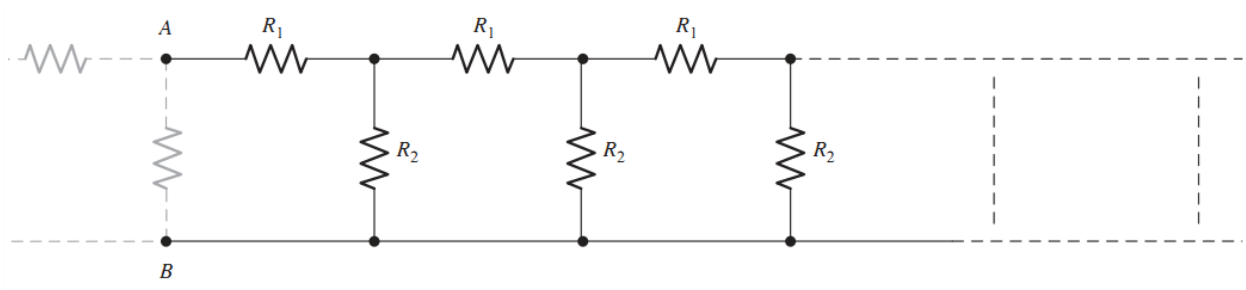
(b) The power dissipated also takes the form $I^2\,R$. Both bulbs now have the same current I, so if Bulb 2 has twice the resistance, as we found in Part (a), then it is twice as bright - the opposite of the case in Part (a). In series, the larger resistor is brighter.

We can also compare the total power dissipated in each case. If the resistances are R and $2\,R$, then in Part (a) the total power dissipated is $\frac{V^2}{R} + \frac{V^2}{2\,R} = \frac{3\,V^2}{2\,R}$. In Part (b) the total power is $I^2\,R + I^2(2\,R) = 3\,I^2\,R$, where $I = \frac{V}{3\,R}$. So the power is $\frac{V^2}{3\,R}$. This is $\frac{2}{9}$ of the power in Part (a). In units of $\frac{V^2}{R}$, the power in Part (a) are 1 and $\frac{1}{2}$, while in Part (b) they are $\frac{1}{9}$ and $\frac{2}{9}$.

Attenuator Chain

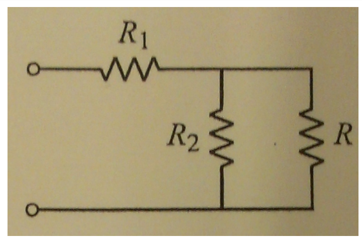
Example
(a) Find the equivalent resistance between terminals A and B in the infinite ladder of resistors shown below.
Hint: Call the input resistance R, and note that it will not be changed by adding a new set of resistors to the front end of the chain to make it one unit longer.
(b) Show that, if voltage V_0 is applied at the input to such a chain, the voltage at successive nodes decreases in a geometric series. What should the ratio of the resistors be so that the ladder is an attenuator that halves the voltage at every step?
(c) Obviously a truly infinite ladder would not be practical. Can you suggest a way to terminate it after a few sections without introducing any error in its attenuation?

Out[ ]=



Solution

(a) If R is the effective resistance of the infinite chain, then the chain is equivalent to the circuit shown below. Thus,



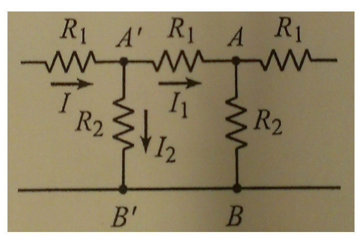
$$R = R_1 + \frac{R_2\,R}{R_2 + R} \tag{29}$$

which implies

$$R = \frac{R_1 + \left(R_1^2 + 4\,R_1\,R_2\right)^{1/2}}{2} \tag{30}$$

where we have chosen the positive root.

(b) To demonstrate the stated geometric series result, consider four points A, A', B, B' that form a square somewhere within the circuit.



Given the voltage V' between A' and B', what is the voltage V between A and B? Let the current flowing towards A' be I, so that (using the effective resistance R of the infinite chain) the current splits into

$$I_1 = \frac{R_2}{R + R_2}\,I \tag{31}$$

$$I_2 = \frac{R}{R + R_2}\,I \tag{32}$$

What will be the current between A and B? By symmetry, it must be $\frac{I_1\,I_2}{I}$, so that

$$\frac{V}{V'} = \frac{I_1}{I} = \frac{R_2}{R_2 + R} \tag{33}$$

(As a quick check, you can make sure that the voltage around the square loop going from A' to A to B to B' is zero, as it must be.) This result is independent of where along the chain we pick the adjacent nodes, so the voltages across successive nodes decrease in a geometric series.

If we want $\frac{V}{V'} = \frac{1}{2}$, then we must have $R = R_2$. Equation (30) then yields

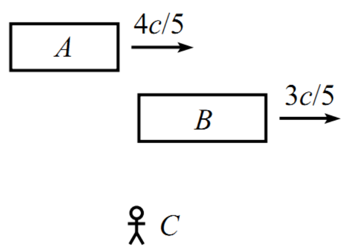
$$2R_2 - R_1 = (R_1^2 + 4R_1R_2)^{1/2} \tag{34}$$
$$(2R_2 - R_1)^2 = R_1^2 + 4R_1R_2 \tag{35}$$
$$R_2 = 2R_1 \tag{36}$$

If we instead wanted $\frac{V}{V'} \approx 1$ (that is, the voltage hardly decreases), then we need $R \ll R_2$, which implies $R_1 \ll R_2$. On the other hand, if we want $\frac{V}{V'} \ll 1$ (that is, the voltage decreases quickly), then we need $R \gg R_2$, which implies $R_1 \gg R_2$. These results make intuitive sense.

(c) To terminate the ladder after any section, without changing its resistance from that of the infinite chain, we can simply connect a single resistor R given by Equation (30) in parallel with the last R_2, because this R mimics the rest of the infinite chain. □

Special Relativity

The following problems all deal with the setup shown below. Two trains, A and B, each have proper length L and move in the same direction. A's speed is $\frac{4}{5}c$, and B's speed is $\frac{3}{5}c$. A starts behind B.



We define the following two events,

$$\begin{aligned}&\text{Event } E_1: \text{"The front of } A \text{ passing the back of} B\text{"}\\ &\text{Event } E_2: \text{"The back of } A \text{ passing the front of} B\text{"}\end{aligned} \tag{37}$$

Length Contraction/Time Dilation

Example

What is the difference in time Δt_C and space Δx_C between E_1 and E_2, as viewed by person C on the ground?

Solution

Relative to C on the ground, the γ factors associated with A and B are

$$\gamma_A^{(C\text{ frame})} = \frac{1}{\left(1-\left(\frac{4}{5}\right)^2\right)^{1/2}} = \frac{5}{3} \tag{38}$$
$$\gamma_B^{(C\text{ frame})} = \frac{1}{\left(1-\left(\frac{3}{5}\right)^2\right)^{1/2}} = \frac{5}{4} \tag{39}$$

Therefore, their lengths in the ground frame are

$$L_A^{(C\text{ frame})} = \frac{L}{\gamma_A^{(C\text{ frame})}} = \frac{3}{5}L \tag{40}$$
$$L_B^{(C\text{ frame})} = \frac{L}{\gamma_B^{(C\text{ frame})}} = \frac{4}{5}L \tag{41}$$

While overtaking B, A must travel farther than B, by an excess distance equal to the sum of the lengths of the trains, which is $L_A^{(C\text{ frame})} + L_B^{(C\text{ frame})} = \frac{3}{5}L + \frac{4}{5}L = \frac{7}{5}L$. The relative speed of the two trains (as viewed by C on the ground) is the difference of the speeds, which is $\frac{c}{5}$. The total time is therefore

$$\Delta t_C = \frac{\frac{7L}{5}}{\frac{c}{5}} = \frac{7L}{c} \tag{42}$$

The distance between both events equals the distance that A travels minus the length of A (since E_1 happens at the front of A and E_2 happens at the rear of A). Therefore

$$\Delta x_C = \Delta t_C\left(\frac{4}{5}c\right) - \frac{3}{5}L = 5L \tag{43}$$

for the two events. □

Velocity Addition

Example

1. Find Δt_B, Δx_B between E_1 and E_2, as viewed by a stationary observer in B's frame? Repeat for Δt_A, Δx_A for a stationary observer in A's frame.
2. Person D walks at constant speed from the back of train B to its front, such that he coincides with both events E_1 and E_2. Compute Δt_D, Δx_D.

Solution

1. In B's reference frame, A moves at a slower speed found by the velocity addition formula

$$u_A^{(B\text{ frame})} = \frac{\frac{4}{5}c - \frac{3}{5}c}{1-\frac{4}{5}\frac{3}{5}} = \frac{5}{13}c \tag{44}$$

with an associated γ factor

$$\gamma_A^{(B\text{ frame})} = \frac{1}{\left(1-\left(\frac{5}{13}\right)^2\right)^{1/2}} = \frac{13}{12} \tag{45}$$

which implies that A has a length

$$L_A^{(B\text{ frame})} = \frac{L}{\gamma_A^{(B\text{ frame})}} = \frac{12}{13}L \tag{46}$$

while B has its rest length

$$L_B^{(B\text{ frame})} = L \tag{47}$$

Therefore the time between events E_1 and E_2 equals

$$\Delta t_B = \frac{L_A^{(B\text{ frame})} + L_B^{(B\text{ frame})}}{u_A^{(B\text{ frame})}} = \frac{\frac{25}{13}L}{\frac{5}{13}c} = \frac{5L}{c} \tag{48}$$

The distance between the two events is simply

$$\Delta x_B = L \tag{49}$$

It is straightforward to repeat all of these calculations in A's frame. Everything will be symmetric (except with everything now moving in the opposite direction), so that

$$\Delta t_A = \frac{5L}{c} \tag{50}$$

$$\Delta x_A = -L \tag{51}$$

2. In B's frame, D must walk with speed $\frac{\Delta x_B}{\Delta t_B} = \frac{1}{5}c$ in order to coincides with both events. In D's reference frame, A will have velocity

$$u_A^{(D\text{ frame})} = \frac{\frac{5}{13}c - \frac{1}{5}c}{1-\frac{5}{13}\frac{1}{5}} = \frac{1}{5}c \tag{52}$$

and B will have velocity

$$u_B^{(D\text{ frame})} = \frac{0-\frac{1}{5}c}{1-0\times\frac{1}{5}} = -\frac{1}{5}c \tag{53}$$

In hindsight, it makes sense that $u_A^{(D\text{ frame})} = -u_B^{(D\text{ frame})}$ because of the symmetry of the problem (i.e. that D coincides with both events and that A and B both have proper length L). The γ factor and length of A will be

$$\gamma_A^{(D\text{ frame})} = \frac{1}{\left(1-\left(\frac{1}{5}\right)^2\right)^{1/2}} = \frac{5}{2\sqrt{6}} \tag{54}$$

$$L_A^{(D\text{ frame})} = \frac{L}{\gamma_A^{(D\text{ frame})}} = \frac{2\sqrt{6}}{5} L \tag{55}$$

and, by symmetry, this must equal the length of B,

$$L_B^{(D\text{ frame})} = \frac{2\sqrt{6}}{5} L \tag{56}$$

Therefore,

$$\Delta t_D = \frac{L_A^{(D\text{ frame})}+L_B^{(D\text{ frame})}}{u_A^{(D\text{ frame})}-u_B^{(D\text{ frame})}} = \frac{\frac{2\sqrt{6}}{5} L}{\frac{1}{5} c} = \frac{2 L\sqrt{6}}{c} \tag{57}$$

Lastly, since D coincides with both events,

$$\Delta x_B = 0 \tag{58}$$

Note: There are several double checks we can perform. For example, the speed of D with respect to the ground frame (i.e. C frame) can be obtained by relativistically adding $\frac{3}{5}c$ and $\frac{1}{5}c$ or subtracting $\frac{1}{5}c$ from $\frac{4}{5}c$. These both give the same answer, namely $\frac{5}{7}c$, as they must. The γ factor between D and the ground is therefore

$$\gamma_D^{(C\text{ frame})} = \frac{1}{\left(1-\left(\frac{5}{7}\right)^2\right)^{1/2}} = \frac{7}{2\sqrt{6}} \tag{59}$$

We can now use time dilation to say that someone on the ground sees the overtaking take a time of $\Delta t_C = \Delta t_D\, \gamma_D^{(C\text{ frame})} = \left(\frac{2L\sqrt{6}}{c}\right)\left(\frac{7}{2\sqrt{6}}\right) = \frac{7L}{c}$, in agreement with equation (42) from the previous problem.

Likewise, the γ factor between D and either train equals

$$\gamma_D^{(A\text{ frame})} = \gamma_D^{(B\text{ frame})} = \frac{1}{\left(1-\left(\frac{1}{5}\right)^2\right)^{1/2}} = \frac{5}{2\sqrt{6}} \tag{60}$$

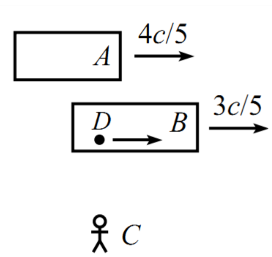
Therefore, the time between events as viewed in A or B equals $\Delta t_A = \Delta t_B = \Delta t_D\, \gamma_D^{(A\text{ frame})} = \left(\frac{2L\sqrt{6}}{c}\right)\left(\frac{5}{2\sqrt{6}}\right) = \frac{5L}{c}$ in agreement with equations (48) and (50) above.

Note that we *cannot* use simple time dilation to relate the ground to A or B, because the two events don't happen at the <u>same place</u> in the train frames! But since both events happen at the same place in D's frame, namely right at D, we can indeed use time dilation to go from D's frame to any other frame. □

Lorentz Transformations

<u>Example</u>

Verify that the values of Δx and Δt found in the above problems satisfy the Lorentz transformations between the six pairs of frames, namely AB, AC, AD, BC, BD, CD.



<u>Solution</u>

Gathering together all of the values found above,

	A	B	C	D
Δt	$\frac{5L}{c}$	$\frac{5L}{c}$	$\frac{7L}{c}$	$\frac{2L\sqrt{6}}{c}$
Δx	$-L$	L	$5L$	0

We also list the various velocities and γ factors from the above examples between each pair of frames

	AB	AC	AD	BC	BD	CD
v	$\frac{5c}{13}$	$\frac{4c}{5}$	$\frac{c}{5}$	$\frac{3c}{5}$	$\frac{c}{5}$	$\frac{5c}{7}$
γ	$\frac{13}{12}$	$\frac{5}{3}$	$\frac{5}{2\sqrt{6}}$	$\frac{5}{4}$	$\frac{5}{2\sqrt{6}}$	$\frac{7}{2\sqrt{6}}$

The Lorentz transformations are

$$\Delta x = \gamma\,(\Delta x' + v\,\Delta t) \tag{61}$$
$$\Delta t = \gamma\left(\Delta t' + \tfrac{v}{c^2}\,\Delta x'\right) \tag{62}$$

For each of the six pairs, we'll transform from the faster frame to the slower frame. This means that the coordinates of the faster frame will be on the right-hand side of the Lorentz transformations. The sign on the right-hand side of the Lorentz transformations will therefore always be a "+". In the AB case, for example, we will write, "Frames B and A," in that order, to signify that the B coordinates are on the left-hand side, and the A coordinates are on
the right-hand side. We'll simply list the Lorentz transformations for the six cases, and you can check that they do indeed all work out.

Frames B and A

$$\begin{aligned} L &= \tfrac{13}{12}\left(-L + \left(\tfrac{5c}{13}\right)\left(\tfrac{5L}{c}\right)\right) \\ \frac{5L}{c} &= \frac{13}{12}\left(\frac{5L}{c} + \frac{\left(\frac{5c}{13}\right)(-L)}{c^2}\right) \end{aligned} \tag{63}$$

Frames C and A

$$\begin{aligned} 5L &= \tfrac{5}{3}\left(-L + \left(\tfrac{4c}{5}\right)\left(\tfrac{5L}{c}\right)\right) \\ \frac{7L}{c} &= \frac{5}{3}\left(\frac{5L}{c} + \frac{\left(\frac{4c}{5}\right)(-L)}{c^2}\right) \end{aligned} \tag{64}$$

Frames D and A

$$\begin{aligned} 0 &= \frac{5}{2\sqrt{6}}\left(-L + \left(\tfrac{c}{5}\right)\left(\tfrac{5L}{c}\right)\right) \\ \frac{2L\sqrt{6}}{c} &= \frac{5}{2\sqrt{6}}\left(\frac{5L}{c} + \frac{\left(\frac{c}{5}\right)(-L)}{c^2}\right) \end{aligned} \tag{65}$$

Frames C and B

$$\begin{aligned} 5L &= \tfrac{5}{4}\left(L + \left(\tfrac{3c}{5}\right)\left(\tfrac{5L}{c}\right)\right) \\ \frac{7L}{c} &= \frac{5}{4}\left(\frac{5L}{c} + \frac{\left(\frac{3c}{5}\right)(L)}{c^2}\right) \end{aligned} \tag{66}$$

Frames B and D

$$\begin{aligned} L &= \frac{5}{2\sqrt{6}}\left(0 + \left(\tfrac{c}{5}\right)\left(\tfrac{2L\sqrt{6}}{c}\right)\right) \\ \frac{5L}{c} &= \frac{5}{2\sqrt{6}}\left(\frac{2L\sqrt{6}}{c} + \frac{\left(\frac{c}{5}\right)(0)}{c^2}\right) \end{aligned} \tag{67}$$

Frames *C* and *D*

$$\begin{aligned} 5L &= \frac{7}{2\sqrt{6}}\left(0+\left(\frac{5c}{7}\right)\left(\frac{2L\sqrt{6}}{c}\right)\right) \\ \frac{7L}{c} &= \frac{7}{2\sqrt{6}}\left(\frac{2L\sqrt{6}}{c}+\frac{\left(\frac{5c}{7}\right)(0)}{c^2}\right) \end{aligned} \tag{68}$$

Closing Remarks

And so ends your second term at Caltech! Take a second to admire the amount of effort and learning during the past ten weeks. We started the term with two postulates in special relativity (all inertial reference frames are "equivalent," and the speed of light is the same in all such frames) and two postulates in electricity (Coulomb's Law and the Principle of Superposition), and we have now built up to problems involving people walking on trains moving near the speed of light, bizarre but beautiful geometries, and circuits with infinite resistors. Amazing!

I hope that these problems leave you with more than the sense that physics is about applying the right equation or following some recipe. At its core, physics is about building intuition - about checking limiting cases, using symmetry, solving problems in multiple ways - so that you do not simply end up with an answer, but rather end up with the confidence to know that your answer is correct.

It has been an unbelievable pleasure and privilege to have taught Ph 1a and Ph 1b at Caltech throughout my PhD. I have met and interacted with so many bright young that already shine so brightly that I cannot wait to see how you will rock the world!

Sincerely,

Tal Einav

2019-03-15

Mathematica Initialization